Differentiaal Vergelijkingen In de Aardwetenschappen

PDEs - chapt 13 - wave equation

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The wave equation (1)

Wave equation in $\mathbb{R}^2$

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \Delta u$$

where

- $u = u(x, y, t)$ transverse displacement
- v = wave propagation speed

Drums

audio

Vibrating drum (1)

- Vibrating circular drum : Boundary Value Problem
- PDE : Wave equation

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

- Domain

$$D = (x, y, t)|x^2 + y^2 < 1, t > 0$$

- Boundary values

$$u(x, y, t) = 0 \quad \text{for} \quad x^2 + y^2 = 1, t > 0,$$
$$u(x, y, 0) = f(x, y)$$

Vibrating drum (2)

Circular drum $\Rightarrow$ Polar Coordinates

$$x = r\ \cos\theta, \qquad y = r\ \sin\theta$$

$$r \geq 0, \quad 0 \leq \theta \leq 2\pi$$

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Laplace operator in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Vibrating drum (3)

Wave equation for $u = u(r, \theta, t)$ in polar coordinates

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Strong (temporary) assumption : Radially symmetric vibrations $u = u(r, t)$

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Boundary conditions for vibrating drum

$$u(1, t) = 0, \;\; t \geq 0$$

$$u(r, 0) = f(r), \;\; 0 \leq r \leq 1$$

Separation of variables (1)

Assume

$$u(r,t) = R(r)T(t)$$

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Wave equation for u = RT

$$\frac{1}{v^2}\frac{\partial^2(RT)}{\partial t^2} = \frac{\partial^2(RT)}{\partial r^2} + \frac{1}{r}\frac{\partial(RT)}{\partial r}$$

$$\Rightarrow \qquad \frac{1}{v^2}R\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 R}{\partial r^2}T + \frac{1}{r}\frac{\partial R}{\partial r}T$$

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Let us define

$$R' = \frac{\partial R}{\partial r} \qquad R'' = \frac{\partial^2 R}{\partial r^2} \qquad T'' = \frac{\partial^2 T}{\partial t^2}$$

then we have

$$\Rightarrow \qquad \frac{1}{v^2}RT'' = R''T + \frac{1}{r}R'T$$

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$$\Rightarrow \qquad \frac{1}{v^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R}$$

Separation of variables (2)

We introduce the separation constant λ as follows :

$$\frac{1}{v^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$

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Eq.(1) writes

$$T'' + \lambda^2 v^2 T = 0$$

so that

$$T(t) = c_1 \cos(\lambda v t) + c_2 \sin(\lambda v t)$$

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Change of variable $s = \lambda r$, $R(r) = R(s/\lambda) = \tilde{R}(s)$. (Note that the b.c. $R(1) = 0$ now writes $\tilde{R}(\lambda) = 0$)
We finally get :

$$s^2 \tilde{R}'' + s\tilde{R}' + s^2 \tilde{R} = 0$$

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This is Bessel's differential equation.

- Friedrich Wilhelm Bessel (22 July 1784 17 March 1846)
- German astronomer, mathematician, physicist and geodesist.
- first astronomer who determined reliable values for the distance from the sun to another star by the method of parallax.
- A special type of mathematical functions were named Bessel functions after his death, though they had originally been discovered by Bernoulli

Bessel functions (1)

Bessel functions, first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, are the canonical solutions *y*(*x*) of **Bessel's differential equation**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

for an arbitrary complex number α (the order of the Bessel function). Although α and −α produce the same differential equation for real α, it is conventional to define different Bessel functions for these two values in such a way that the Bessel functions are mostly smooth functions of α.

The most important cases are for α an integer or half-integer. Bessel functions for integer α are also known as **cylinder functions** or the **cylindrical harmonics** because they appear in the solution to Laplace's equation in cylindrical coordinates. **Spherical Bessel functions** with half-integer α are obtained when the Helmholtz equation is solved in spherical coordinates.

Bessel functions (2)

Because this is a second-order differential equation, there must be two linearly independent solutions. Depending upon the circumstances, however, various formulations of these solutions are convenient. Different variations are summarized in the table below, and described in the following sections.

Type	First kind	Second kind
Bessel functions	J_α	Y_α
modified Bessel functions	I_α	K_α
Hankel functions	$H_\alpha^{(1)} = J_\alpha + iY_\alpha$	$H_\alpha^{(2)} = J_\alpha - iY_\alpha$
Spherical Bessel functions	j_n	y_n
Spherical Hankel functions	$h_n^{(1)} = j_n + iy_n$	$h_n^{(2)} = j_n - iy_n$

Bessel functions (3)

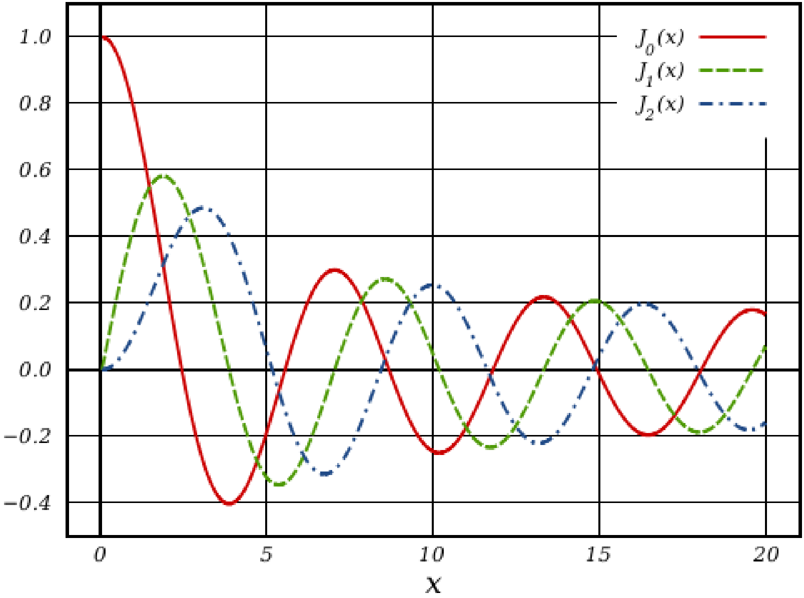
Bessel functions of the first kind: J_α [edit]

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation that are finite at the origin ($x = 0$) for integer or positive α, and diverge as x approaches zero for negative non-integer α. It is possible to define the function by its series expansion around $x = 0$, which can be found by applying the Frobenius method to Bessel's equation:[1]

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

where $\Gamma(z)$ is the gamma function, a shifted generalization of the factorial function to non-integer values. The Bessel function of the first kind is an entire function if α is an integer, otherwise it is a multivalued function with singularity at zero. The graphs of Bessel functions look roughly like oscillating sine or cosine functions that decay proportionally to $1/\sqrt{x}$ (see also their asymptotic forms below), although their roots are not generally periodic, except asymptotically for large x. (The series indicates that $-J_1(x)$ is the derivative of $J_0(x)$, much like $-\sin(x)$ is the derivative of $\cos(x)$; more generally, the derivative of $J_n(x)$ can be expressed in terms of $J_{n\pm1}(x)$ by the identities below.)

Bessel functions (4)



Solution to the PDE (1)

A solution to

$$s^2 \tilde{R}'' + s\tilde{R}' + s^2 \tilde{R} = 0$$

is then the first kind of Bessel function $J_0(s)$, or

$$R(r) = J_0(\lambda r)$$

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$$R(r) = J_0(\lambda r)$$

The eigenvalue λ is obtained from the boundary condition

$$R(1) = J_0(\lambda) = 0$$

i.e. λ is a zero of the J_0 Bessel function.

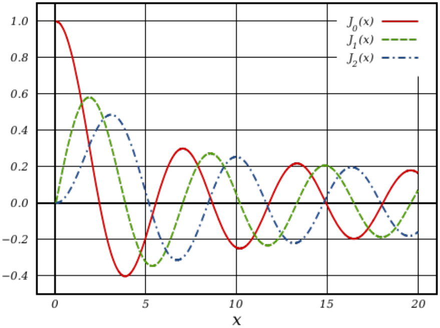
Solution to the PDE (2)

Finally the solution to the PDE can be written :

$$u(r,t) = R(r)T(t) = J_0(\lambda r)\left[c_1\cos(\lambda v t) + c_2\sin(\lambda v t)\right]$$

where λ_{0k} is the k-th zero of $J_0(\lambda) = 0$.

Solution to the PDE (3)



k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

film

Non radially symmetric solutions

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Separation of variables :

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

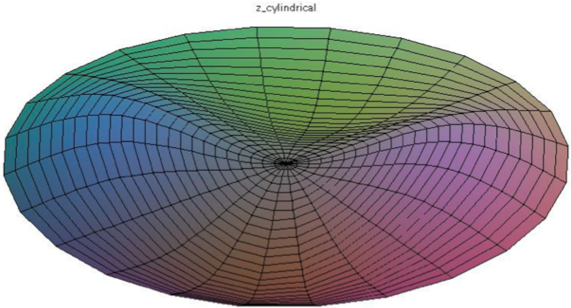
Three ODEs, two eigenvalue problems (for R and Θ).
Solutions with eigenvalues n, λ :

- $\Theta = \cos(n\theta)$, $n = 0{,}1{,}2{,}3,$
- $R = J_n(\lambda r)$, Bessel function of order n
- $T = cos(\lambda vt)$
- $u(r,\theta,t) = J_n(\lambda r)\cos(n\theta)\cos(\lambda vt)$

Eigenvalue $\lambda = \lambda_{nk}$ is the k-th zero of $J_n(\lambda) = 0$

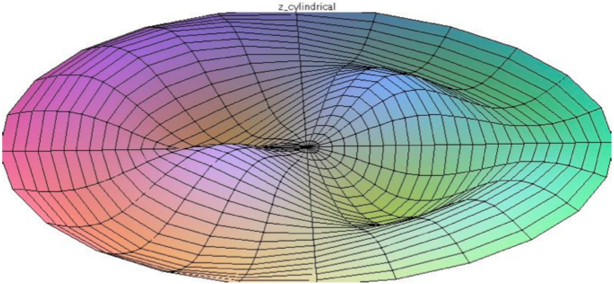
Solution to PDE (1)

$$u(r,\theta) = J_2(\lambda_{21} r)\cos(2\theta)$$



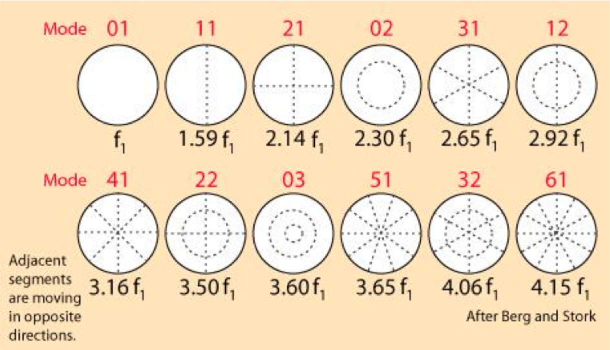
Solution to PDE (2)

$$u(r, \theta) = J_2(\lambda_{22} r) \cos(2\theta)$$



$$u(r, \theta) = 0 \text{ if } J_2(\lambda_{22} r) = 0 \text{ or } \cos(2\theta) = 0$$

Solution to PDE (3)



Superposition

For arbitrary initial value

$$u(r,\theta) = f(r,\theta)$$

vibration of drum = Superposition of modes

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} c_{kn} J_n(\lambda_{nk} r)\cos(n\theta)\cos(\lambda_{nk} vt) + \cdots$$

Coefficients c_{kn} are coefficients in "Fourier-Bessel" series of f.