Differentiaal Vergelijkingen In de Aardwetenschappen

PDEs - chapt 13

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January 2015

Introduction

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- Schrödinger equation

$$-\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 \Psi + V\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

The Laplace equation

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- Laplace Equation describes steady state situations such as :
  - steady state temperature distributions
  - steady state stress distributions
  - steady state potential distributions (it is also called the potential equation
  - steady state flows, for example in a cylinder, around a corner

The Laplace equation (2)

In cartesian coordinates :

- 1D :

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- 3D :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The Laplace equation (3)

Example 1 : (Maxwell's equations)
The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

with

- $\mathbf{E}$: electric field
- ρ : charge density
- ϵ_0 : vacuum permittivity (=8.854187817620... $\times 10^{-12}$ Farads per metre)

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In a charge-free region of space ($\rho = 0$), the potential is therefore related to the charge density by Laplace equation

$$\nabla^2 V = 0$$

The Laplace equation (4)

Example 2 :

The heat transport equation writes :

$$\rho c_p \left(\frac{\partial T}{\partial t} + \underbrace{\mathbf{v} \cdot \boldsymbol{\nabla} T}_{adv.} \right) = \underbrace{\boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)}_{diff.} + \underbrace{H}_{prod.}$$

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If $k = constant$, then it is equivalent to solve

$$\boldsymbol{\nabla}^2 T = 0$$

Steady state temperature in 1D

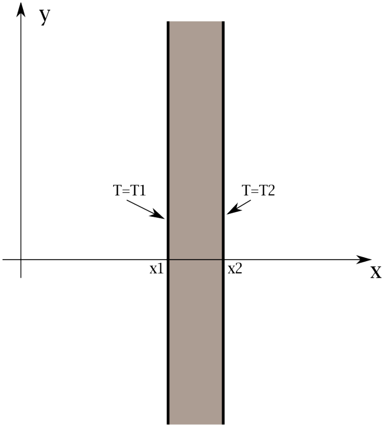

plate is infinite in y and z direction : $\frac{\partial}{\partial y} \rightarrow 0$, $\rightarrow \frac{\partial}{\partial z} \rightarrow 0$

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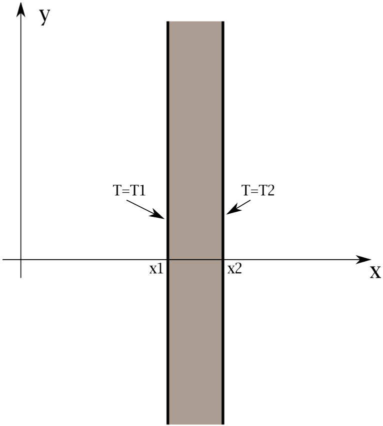

plate is infinite in y and z direction : $\frac{\partial}{\partial y} \to 0$, $\to \frac{\partial}{\partial z} \to 0$
$\Rightarrow$ T is a function of x only.

Steady state temperature in 1D (2)

We have to solve :

$$\frac{d^2 T}{dx^2} = 0$$

with boundary conditions :

$$T(x_1) = T1 \qquad T(x_2) = T2$$

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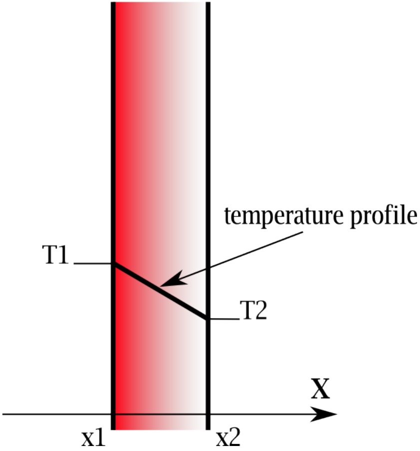
$$\begin{aligned} T(x_1) = T_1 &= ax_1 + b \\ T(x_2) = T_2 &= ax_2 + b \end{aligned}$$

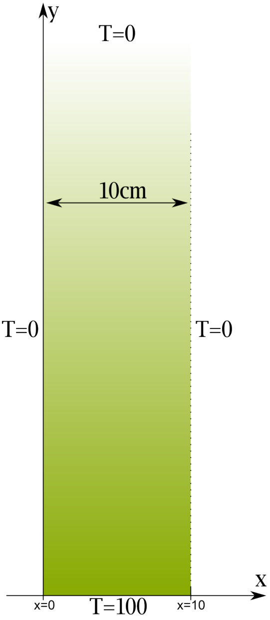
leading to $a = \frac{T_2 - T_1}{x_2 - x1}$ and $b = T_1 - \frac{T_2 - T_1}{x_2 - x1} x_1$ and finally

$$T(x) = \frac{T_2 - T_1}{x_2 - x1}(x - x_1) + T_1$$

Steady state temperature in 1D (3)



Steady-state temperature in a rectangular plate



Steady-state temperature in a rectangular plate (2)

The temperature satisfies the 2D Laplace equation inside the plate :

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We could try to solve the equation by using a tentative solution of the form :

$$T(x, y) = \theta(x)\Phi(y) \tag{2}$$

We do not *know* the solution is of this form.

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We substitute (2) into (1) and obtain :

$$\Phi \frac{\partial^2 \theta}{\partial x^2} + \theta \frac{\partial^2 \Phi}{\partial y^2} = 0$$

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$$\Phi \frac{\partial^2 \theta}{\partial x^2} + \theta \frac{\partial^2 \Phi}{\partial y^2} = 0$$

Dividing by $\theta\Phi$ gives :

$$\frac{1}{\theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial y^2} = 0$$

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Separation of variables : we say that each term is a constant because the first term is a function of x only and the second a function of y only.
We then write

$$\frac{1}{\theta}\frac{\partial^2\theta}{\partial x^2} = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial y^2} = -k^2$$

where k is called the separation constant.

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- The solution to the second one is $\Phi(x) = e^{kx}$ or $\Phi(x) = e^{-kx}$

Steady-state temperature in a rectangular plate (3)

The general solution writes :

$$T(x,y) = \theta(x)\Phi(y) = \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \left\{ \begin{array}{c} e^{ky} \\ e^{-ky} \end{array} \right\}$$

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$$T(x,y) = \sin(kx)\, e^{-ky}$$

We finally use $T = 0$ at $x = 10$ which leads to $10k = n\pi$, i.e. :

$$T(x,y) = \sin(\frac{n\pi x}{10})\, e^{-n\pi y/10}$$

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Problem : the solution does not satisfy $T(x, 0) = 100$!

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Let's find such a combination which satisfies the b.c. at $y = 0$:

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We impose then $T(x,0) = 100$:

$$100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$$

Steady-state temperature in a rectangular plate (5)

$$100 = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{10})$$

This is the Fourier sine series of $f(x) = 100$ with $l = 10$ (chapter 7.9 of Boas).

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$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{10}\int_0^l 100\sin\frac{n\pi x}{10}dx = \begin{cases} 400/n\pi & \text{odd n} \\ 0 & \text{even n} \end{cases}$$

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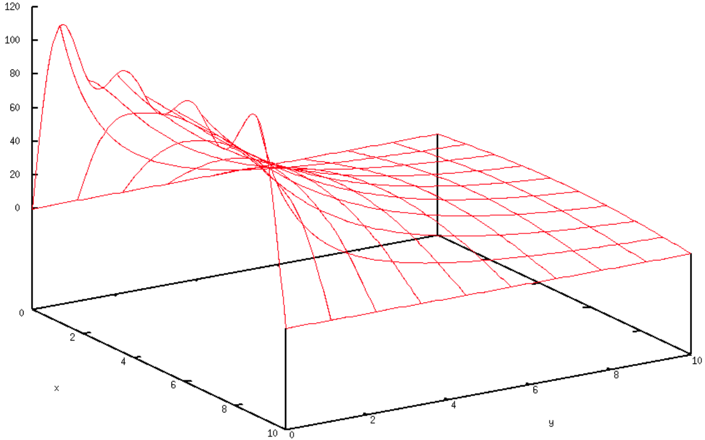
Finally (!) :

$$T(x,y) = \frac{400}{\pi}\left(e^{-\pi y/10} \sin(\frac{\pi x}{10}) + \frac{1}{3}\sin(\frac{3\pi x}{10})\, e^{-3\pi y/10} + \ldots \right)$$

Steady-state temperature in a rectangular plate (6)



(n=1,3,5 and 7)

Numerical solution (1)

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Assume :

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Then the temperature field must verify :

$$\rho c_p \frac{\partial T}{\partial t} = \underbrace{\boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)}_{diff.}$$

And at steady state ($\partial_t = 0$)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) = 0$$

Numerical solution (2)

film

Numerical solution (3)