Differentiaal Vergelijkingen In de Aardwetenschappen

Ordinary Differential Equations, chapter 8, sections 1-4

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new term, definition

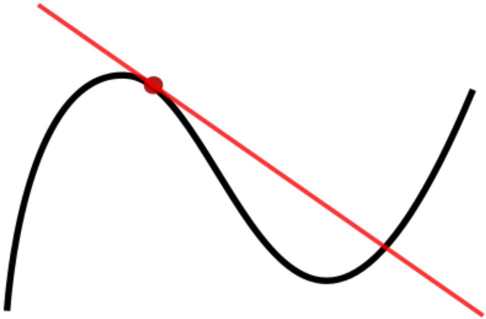
Exercise for werkcollege

Homework

pay attention to this

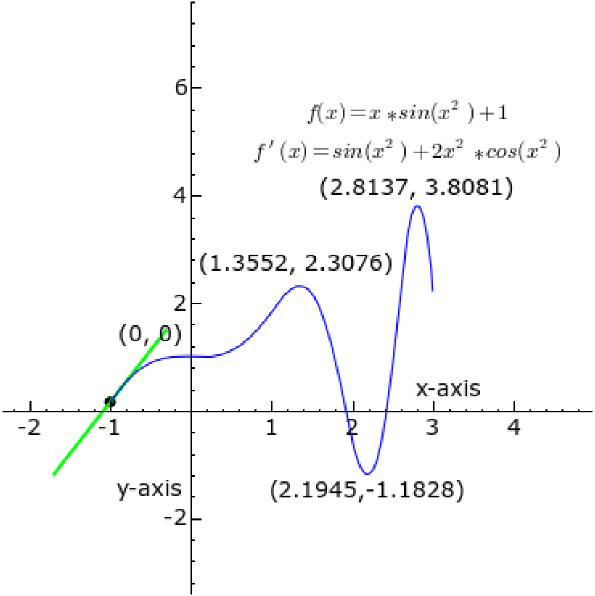
What is a derivative?!?

- In calculus, a branch of mathematics, the derivative is a measure of how a function changes as its input changes.
- The derivative of a function at a chosen input value describes the best linear approximation of the function near that input value.
- For a real-valued function of a single real variable, the derivative at a point equals the slope of the tangent line to the graph of the function at that point.



(The slope of the tangent line is equal to the derivative of the function at the marked point)

What is a derivative ? ! ?



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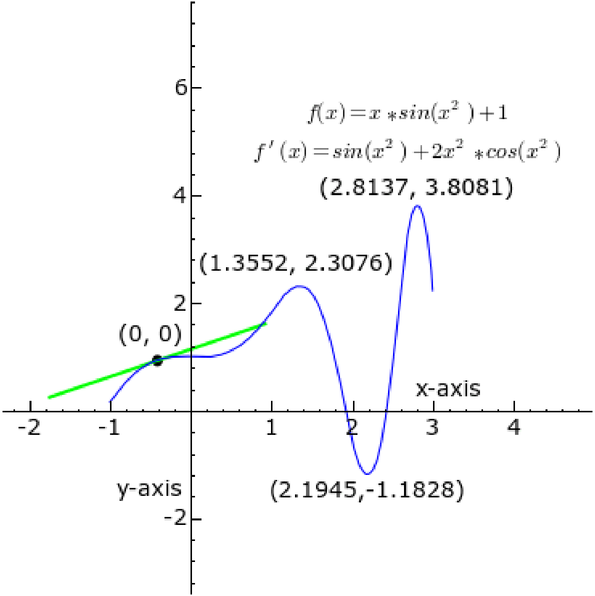

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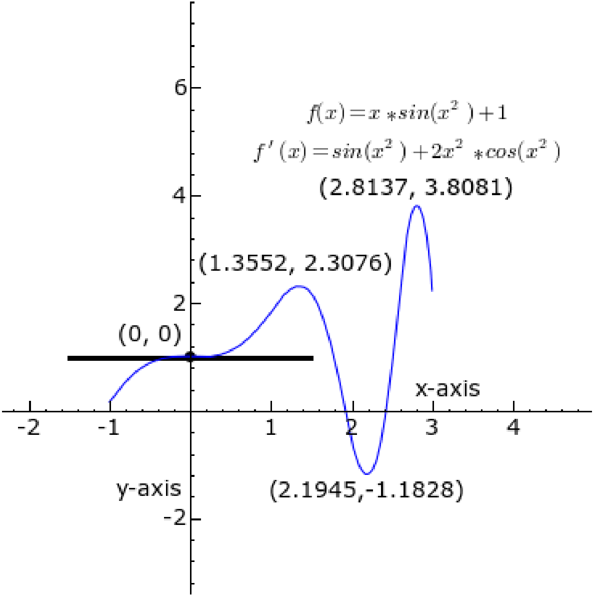

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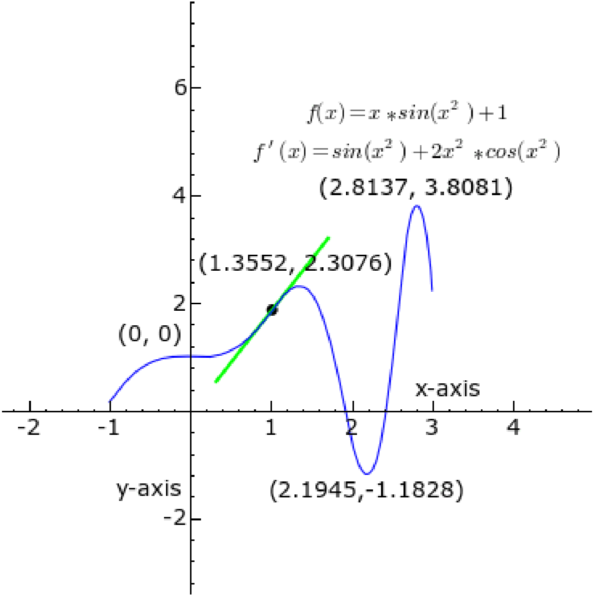

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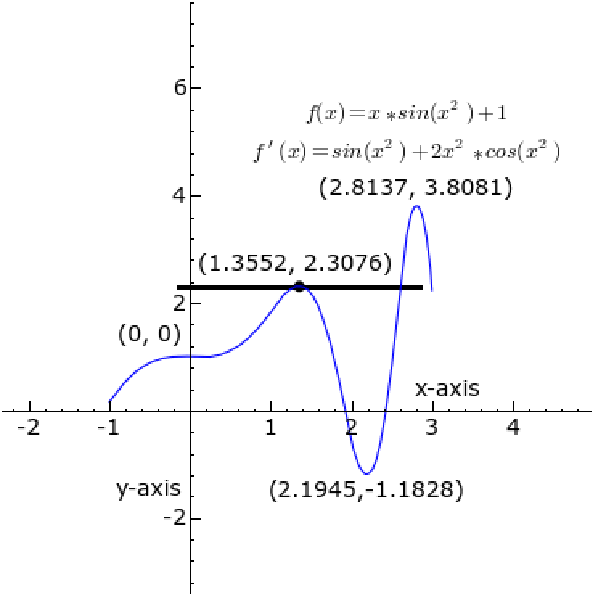

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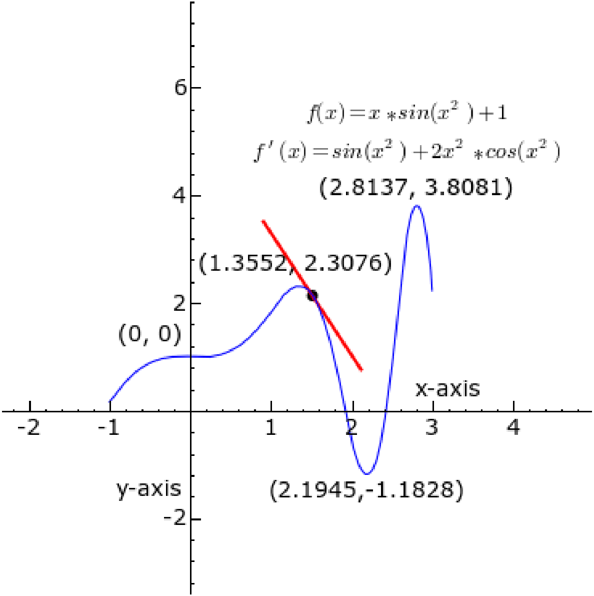

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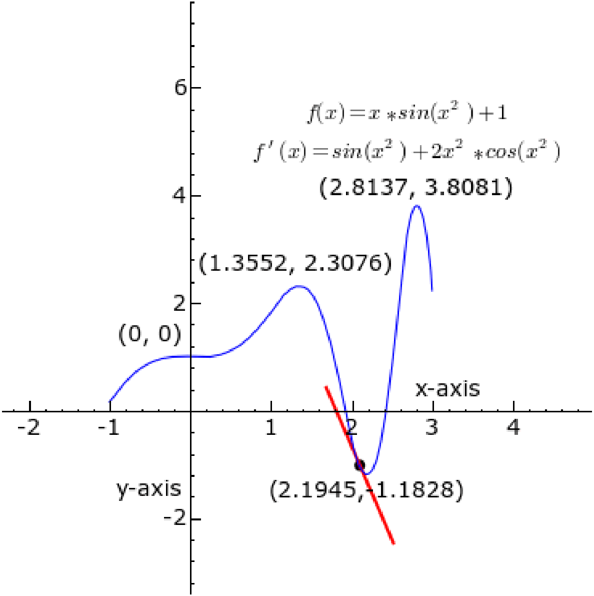

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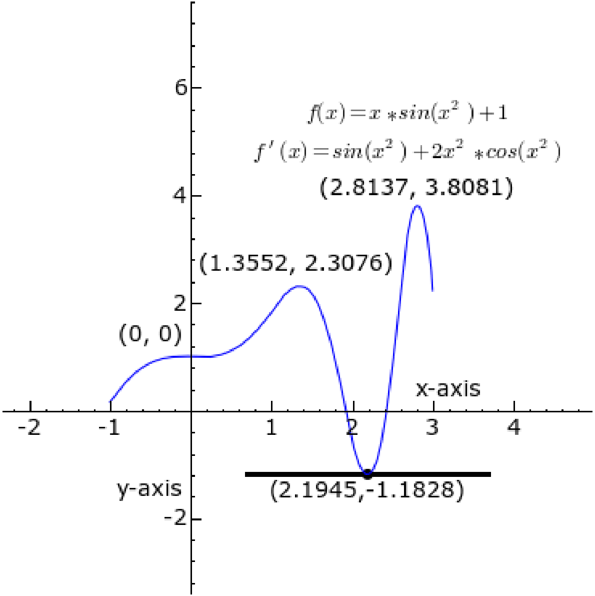

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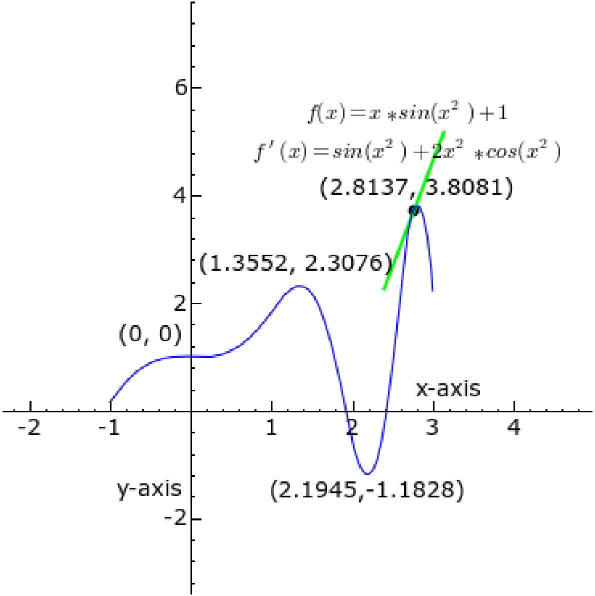

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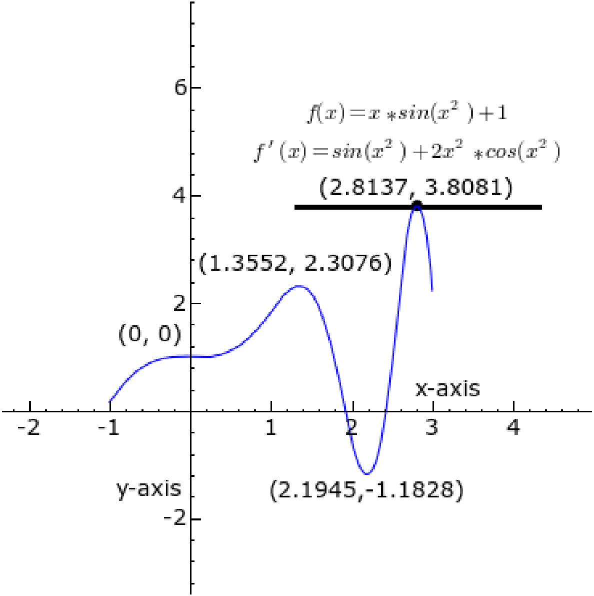

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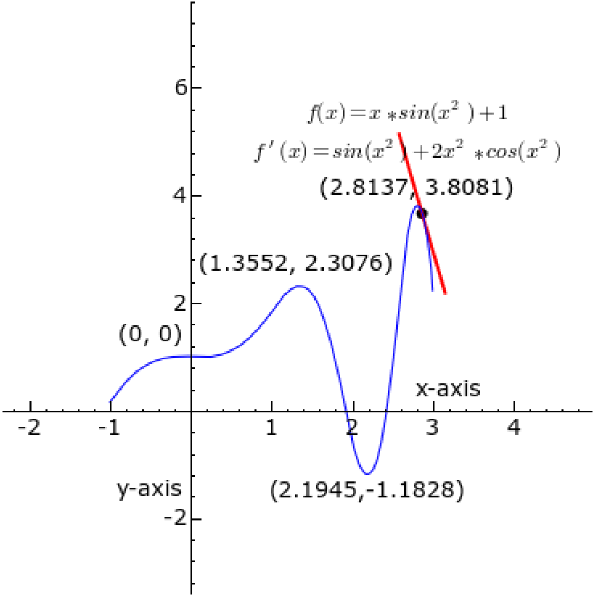

What is a derivative - Leibniz's notation

The first derivative is denoted by

$$\frac{dy}{dx} = \frac{df}{dx}(x) = \frac{d}{dx}f(x)$$

Higher derivatives are expressed using the notation

$$\frac{d^n y}{dx^n}, \quad \frac{d^n f}{dx^n}(x), \text{ or } \frac{d^n}{dx^n}f(x)$$

for the n^{th} derivative of $y = f(x)$ (with respect to x). These are abbreviations for multiple applications of the derivative operator. For example,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

What is a derivative - other notation

- Lagrange's notation

$$(f')' = f'' \quad \text{and} \quad (f'')' = f'''$$

- Newton's notation (time derivatives)

$$\dot{y} \quad \text{and} \quad \ddot{y}$$

- Euler's notation

$$D_x y \quad \text{or} \quad D_x f(x)$$

Derivatives of elementary functions (1)

- Derivatives of powers : if

$$f(x) = x^r,$$

where r is any real number, then

$$f'(x) = rx^{r-1}$$

wherever this function is defined. For example, if $f(x) = x^{1/4}$, then

$$f'(x) = (1/4)x^{-3/4}$$

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- Exponential and logarithmic functions
$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = \ln(a)a^x$$
$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0 \qquad \frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

Derivatives of elementary functions (2)

- Trigonometric functions :

$$\frac{d}{dx}\sin(x) = \cos(x).$$

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

$$\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x).$$

- Inverse trigonometric functions :

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Rules for finding the derivative

- Constant rule : if $f(x)$ is constant, then

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- Chain rule : If $f(x) = h(g(x))$, then

$$f'(x) = h'(g(x)) \cdot g'(x)$$

Spatial derivatives in 1D, 2D, 3D (1)

- in 1D, one space dimension :
$$\frac{\partial}{\partial x}$$
- in 2D, one space dimension :
$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}$$
- in 3D, one space dimension :
$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z}$$

$\Rightarrow$ more concise notation : $\boldsymbol{\nabla}$ ('nabla' operator, or 'gradient')

Spatial derivatives in 1D, 2D, 3D (2)

For instance, in 3D the gradient operator writes :

$$\boldsymbol{\nabla} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \\ \\ \dfrac{\partial}{\partial z} \end{pmatrix}$$

It is therefore a <u>vector</u>.
The temperature gradient (i.e. the spatial derivative of the temperature field) writes :

$$\boldsymbol{\nabla} T = \begin{pmatrix} \dfrac{\partial T}{\partial x} \\ \\ \dfrac{\partial T}{\partial y} \\ \\ \dfrac{\partial T}{\partial z} \end{pmatrix}$$

The gradient operator is applied to a scalar and the result is a vector.

Spatial derivatives in 1D, 2D, 3D (2)

If one now takes the scalar product of the gradient with a vector $\mathbf{v} = (v_x, v_y, v_z)$, one gets a scalar :

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

The $\boldsymbol{\nabla}\cdot$ operator is called the divergent operator. It applies on vectors and gives a scalar.

Spatial derivatives in 1D, 2D, 3D (2)

It is possible to combine both gradient and divergent operator in order to get a second-order operator :

$$\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \\ \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \\ \\ \frac{\partial}{\partial z} \end{pmatrix} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Δ is called the Laplacian operator. It the "divergence of the gradient".
For instance, the Laplacian of the temperature field $T(x, y, z)$ writes :

$$\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Maths, hairstyle and history (continued)

Pierre-Simon, marquis de Laplace (1749-1827),
French mathematician and astronomer

Introduction

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- If it contains partial derivatives, it is called Partial Differential Equation (PDE)

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- If it contains partial derivatives, it is called Partial Differential Equation (PDE)
- Otherwise it is called an Ordinary Differential Equation (ODE)

Example (1)

- Newton's second law. In vector form, it writes :

$$\sum \mathbf{F} = m\mathbf{a}$$

Writing the acceleration $\mathbf{v}$ as $d\mathbf{v}/dt$, where $\mathbf{v}$ is the velocity, we get

$$\sum \mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

This is in fact a set of ODEs (one for each direction in space) :

$$\begin{aligned} F_x &= m\frac{du}{dt} \\ F_y &= m\frac{dv}{dt} \\ F_z &= m\frac{dw}{dt} \end{aligned}$$

where $\mathbf{v} = (u, v, w)$.

Example (2) - heat equation

The rate at which heat Q escapes through a window or from a hot water pipe is proportional to the area A and to the rate of change of temperature with distance in the direction of the flow of heat :

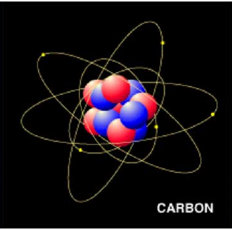
$$\frac{dQ}{dt} = kA\frac{dT}{dx}$$

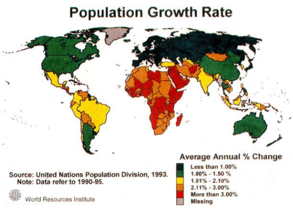
where k is called the coefficient of thermal conductivity and is a property of the material.

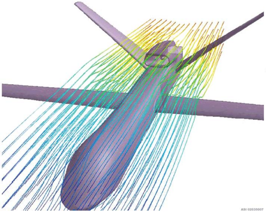
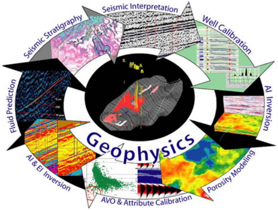
Introduction (2)

⇒ Differential equations are oblivious :

- from astrophysics to quantum mechanics
- from vibrating strings and membranes to population growth
- in fluid mechanics, in solid mechanics, geophysics ...

Introduction (3)

They come in all shapes, colours and sizes :)

$$\begin{aligned}
\partial_t n^\alpha &= -\nabla \mathbf{v} n^\alpha + \nabla\left(\sum_\beta D^{\alpha\beta}\hat{T}\nabla\left(\frac{\hat{\mu}^\beta}{\hat{T}}\right)\right) - \nabla\left(S^\alpha \hat{T}^2 \nabla\left(\frac{1}{\hat{T}}\right)\right) - \nabla\left(\sum_\beta D^{\alpha\beta}\overline{\mathbf{F}}^\beta\right), \\
\partial_t \mathbf{g} &= -\nabla\cdot(\mathbf{g}\mathbf{v}) - \nabla\hat{P} - \nabla\hat{\mathbf{\Pi}} + \sum_\alpha n^\alpha \overline{\mathbf{F}}^\alpha, \\
\partial_t \hat{\epsilon} &= -\nabla\cdot(\mathbf{v}\hat{\epsilon}) - \hat{P}\nabla\cdot\mathbf{v} - \hat{\mathbf{\Pi}} : \nabla\mathbf{v} + \nabla\left(\sum_\alpha S^\alpha \hat{T}^2 \nabla\left(\frac{\hat{\mu}^\alpha}{\hat{T}}\right)\right) - \nabla\left(\hat{\kappa}\hat{T}^2\nabla\left(\frac{1}{\hat{T}}\right)\right) \\
&\quad - \sum_{\alpha\beta}\left[\hat{T}\nabla\left(\frac{\hat{\mu}^\alpha}{\hat{T}}\right) - \overline{\mathbf{F}}^\alpha\right] D^{\alpha\beta}\overline{\mathbf{F}}^\beta - \nabla\left(\sum_\alpha S^\alpha \overline{\mathbf{F}}^\alpha \hat{T}\right) - \sum_\alpha S^\alpha \overline{\mathbf{F}}^\alpha \nabla\hat{T}, \qquad (4.8)
\end{aligned}$$

Smoothed Particle Hydrodynamics model for phase separating fluid mixtures, C. Thieulot, P. Español and L.P.B.M.Janssen, Phys. Rev. E 72, 016714 (2005).

Definitions

- The order of a differential equation is the order of the highest derivative in the equation.

$$y' + xy^2 = 1$$
$$xy' + y = e^x$$

are first-order ODEs.

$$m\frac{dr^2}{dt^2} = -kr$$

is a second-order ODE.

- A linear ODE is one of the form :

$$a_0y + a_1 + y' + a_2y'' + a_3y''' + \ldots = b$$

where the a's and b are either constants or functions of x.

$$y' = \cot y \qquad (\textit{not linear})$$
$$yy' = 1 \qquad (\textit{not linear})$$
$$(y')^2 = xy \qquad (\textit{not linear})$$

Definitions (2)

- A solution of a differential equation (in the variables x and y) is a relation between x and y which, if substituted into the differential equation, gives and indentity.
- Example 1 : the relation $y = \sin x + C$ is a solution of the differential equation $y' = \cos x$.
- Example 2 : The equation $y'' = y$ has solutions $y = e^x$ or $y = e^{-x}$ or $y = Ae^x + Be^{-x}$.

Definitions (3)

- Any *linear* differential equation of order n has a solution containing n independent arbitrary constants, from which all solutions of the differential equation can be obtained by letting the constants have particular values. This solution is called the general solution of the linear differential equation.

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  This may not be true for nonlinear equations.

Example 3

Question : Find the distance which an object falls under gravity in t seconds if it starts from rest

We start with Newton's second law

$$m\mathbf{a} = \mathbf{F}$$

where m is the mass of the object. The gravitational acceleration is $\mathbf{g}$, so that the force F is given by $m\mathbf{g}$.

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Let z be the distance the object has fallen in time t, and let us assume that the movement of the object occurs along the vertical z axis. We then have to solve

$$\frac{d^2z}{dt^2} = g_z$$

Example 3

We integrate one time and get

$$\frac{dz}{dt} = g_z t + C = g_z t + v_0$$

and we integrate a second time :

$$z(t) = \frac{1}{2} g_z t^2 + v_0 t + z + 0$$

Two important things :

- the object starts from rest

$$v_0 = v(t = 0) = 0$$

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Two important things :

- the object starts from rest

$$v_0 = v(t = 0) = 0$$

- $z(t)$ is the distance the object has fallen at time t

$$z_0 = z(t = 0) = 0$$

Finally :

$$z(t) = \frac{1}{2} g_z t^2$$

Example 4 :

Question : Find the solution of $y'' = y$ which passes through the origin and through the point $(\ln 2, 3/4)$.

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We have already verified that $y = Ae^x + Be^{-x}$ is a solution of the differential equation. If the given points satisfy the equation of the curve, then

$$0 = A + B \qquad \frac{3}{4} = Ae^{\ln 2} + Be^{-\ln 2}$$

which leads to write

$$\begin{aligned} A + B &= 0 && (1) \\ 2A + B/2 &= 3/4 && (2) \end{aligned}$$

$\Rightarrow A = -B = 1/2$ and

$$y(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

Definitions (4)

- The given conditions which are to be satisfied by the particular solution are called boundary conditions.

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- When they are conditions at $t = 0$, they may be called initial conditions

Separable equations

When we can write

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$$dy = f(x)dx$$

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$$y' = \frac{dy}{dx} = f(x) \qquad \Rightarrow \qquad dy = f(x)dx \qquad \Rightarrow \qquad y = \int f(x)dx$$

Solving the ODE is carried out by integrating each side of the equation.

Separable equations - Example 1

The rate at which a radioactive substance decays is proportional to the remaining number of atoms. If there are N_0 atoms at $t = 0$, find the number at time t.

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The rate at which a radioactive substance decays is proportional to the remaining number of atoms. If there are N_0 atoms at $t = 0$, find the number at time t.

This translates as

$$\frac{dN}{dt} = -\lambda N$$

where λ is called the decay constant. It also writes :

$$\frac{dN}{N} = -\lambda dt$$

Integrating both sides we get

$$\ln N = -\lambda t + const$$

Since $N(t = 0) = N_0$, then $const = N_0$ and finally

$$N(t) = N_0 e^{-\lambda t}$$

Separable equations - Example 2

Solve the differential equation

$$xy' = y + 1$$

Separable equations - Example 2

Solve the differential equation

$$xy' = y + 1$$

To separate variables we divide both sides by $x(y+1)$:

$$\frac{y'}{y+1} = \frac{1}{x}$$

Recall that $y' = \frac{dy}{dx}$ so that :

$$\frac{dy}{y+1} = \frac{dx}{x}$$

and then

$$\int \frac{1}{y+1} dy = \int \frac{1}{x} dx$$

Carrying out the integration :

$$\ln(y+1) = \ln x + C = \ln x + \ln a = \ln(ax)$$

finally :

$$y + 1 = ax$$

Nonlinear differential equations

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Example : Stokes equation

$$-\boldsymbol{\nabla} p + \mu \Delta \mathbf{v} = \rho \mathbf{g}$$

Often $\mu = fct(T, p, \frac{d\mathbf{v}}{dt})$ and $\rho = fct(T, z)$


Sir George Gabriel Stokes (1819-1903)

Linear first-order equations (1)

- a first-order equation contains y' but no higher derivative.
- a linear first-order equation means one which can be written in the form

$$y' + Py = Q$$

where P and Q are functions of x.

Linear first-order equations (2)

- Let's start with an easier case : $Q = 0$

$$y' + Py = 0$$

This is equivalent to

$$\frac{dy}{dx} = -Py$$

The equation is separable :

$$\frac{dy}{y} = -Pdx$$

$$\ln y = -\int P\ dx + C$$

$$y = e^{-\int P\ dx + C} = Ae^{-\int P\ dx}$$

Let us simplify the notation and write

$$I = \int P\ dx \qquad \text{or,} \qquad \frac{dI}{dx} = P$$

Linear first-order equations (2)

- Let's solve the full ODE : $y' + Py = Q$

We know that $y = Ae^{-I}$ is solution of $y' + Py = 0$.
Let us compute

$$\begin{aligned}
\frac{d}{dx}(y\,e^{I}) &= y'e^{I} + y\frac{d}{dx}e^{I} \\
&= y'e^{I} + ye^{I}\frac{dI}{dx} \\
&= y'e^{I} + ye^{I}P \\
&= (y' + yP)e^{I} \\
&= Qe^{I}
\end{aligned}$$

So

$$ye^{I} = \int Qe^{I}dx + const$$

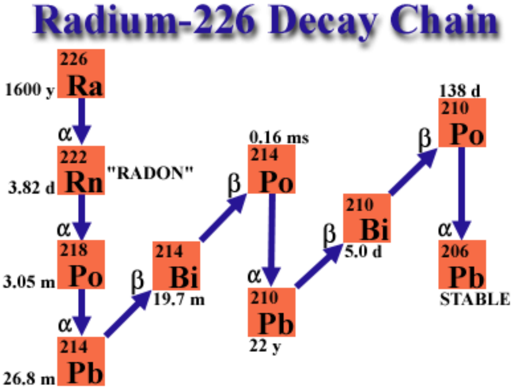
or,

$$y(x) = e^{-I}\left(\int Qe^{I}dx + const\right) \qquad \text{with} \qquad I = \int P\,dx$$

This is the general solution, up to a constant.

Example - A touch of radiometric dating ...

Radium decays to radon which decays to polonium



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Question : If at $t = 0$ a sample is pure radium, how much radon does it contain at time t ?

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$$\text{Let} \quad \begin{aligned} N_0 &= \text{number of radium atoms at } t = 0 \\ N_1 &= \text{number of radium atoms at time } t \\ N_1 &= \text{number of radon atoms at time } t \\ \lambda_1 &= \text{decay constant of radium} \\ \lambda_2 &= \text{decay constant of radon} \end{aligned}$$

We have for radium :

$$\frac{dN_1}{dt} = -\lambda_1 N_1$$

or, as we have seen before : $N_1(t) = N_0 e^{-\lambda_1 t}$.
The rate at which radon is created is the rate at which radium is decaying, i.e. $\lambda_1 N_1$. But radon is decaying at the rate $\lambda_2 N_2$. Hence we have :

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

Example - A touch of radiometric dating ... (3)

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

rewrites

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_1$$

and is of the form

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Example - A touch of radiometric dating ... (3)

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We solve it as follows :

$$I = \int P dt = \int \lambda_2 dt = \lambda_2 t$$

Example - A touch of radiometric dating ... (4)

and using

$$y(x) = e^{-I} \int Qe^{I} dx + const$$

we can write

$$\begin{aligned} N_2(t) &= e^{-\lambda_2 t} \left(\int \lambda_1 N_1 e^{\lambda_2 t} dt + const \right) \\ &= e^{-\lambda_2 t} \left(\int \lambda_1 N_0 e^{\lambda_1 t} e^{\lambda_2 t} dt + const \right) \\ &= e^{-\lambda_2 t} \left(\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + const \right) \end{aligned}$$

valid for $\lambda_1 \neq \lambda_2$.
Finally, $N_2(t=0) = 0$ so that

$$const = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}$$

and

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

Other methods for 1st order equations (1)

The Bernoulli equation

The differential equation

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where P and Q are functions of x is known as the Bernoulli equation.
We make the change of variable $z = y^{1-n}$. Then $z' = (1-n)y^{-n}y'$ and the differential equation reqrites :

$$z' + (1-n)Pz = (1-n)Q$$

This is now a first-order linear equation which we can solve.

Other methods for 1st order equations (2)

Exact equations

Let us recall that the differential of $F(x, y)$ writes :

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Other methods for 1st order equations (2)

Exact equations

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The expression $P(x, y)dx + Q(x, y)dy$ is an exact differential (i.e. a differential of a function F) if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

If this is verified, the solution of $Pdx + Qdy = 0$ is then

$$F(x, y) = const$$

Other methods for 1st order equations (2)

Example :

- The equation $xdy - ydx = 0$ is not exact.

Other methods for 1st order equations (2)

Example :

- The equation $xdy - ydx = 0$ is not exact.
- The equation

$$\frac{1}{x^2}(xdy - ydx) = 0$$

is exact.

$$P = -y/x^2 \qquad Q = \frac{1}{x} \qquad \rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Other methods for 1st order equations (3)

A homogeneous function of x and y of degree n means a function which can be written as $x^n f(y/x)$.
An equation of the form

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If y' can be written as a function of y/x, we make the change of variable $v = y/x$ and solve for v.

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- such second-order equations are of the form

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

Second-order linear equations with constant coeffs. and zero rhs

Example : solve the equation

$$y'' + 5y' + 4y = 0$$

- the coefficients a_0, a_1, a_2 are constant
- the right-hand side is zero

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It is convenient to let D stand for d/dx so that

$$Dy = \frac{dy}{dx} \qquad D^2 y = D \cdot Dy = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2 y}{dx^2} = y''$$

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Expressions involving D, such as $D + 1$ or $D^2 + 5D + 4$ are called differential operators

$$y'' + 5y' + 4y = 0 \qquad \rightarrow \qquad (D^2 + 5D + 4)y = 0$$

Second-order linear equations with constant coeffs. and zero rhs (2)

$$(D^2+5D+4)y=0 \qquad \rightarrow \qquad (D+1)(D+4)y=0 \quad \text{or} \quad (D+4)(D+1)y=0$$

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Let us consider the simpler equations

$$(D+4)y=0 \qquad (D+1)y=0$$

or,

$$\frac{dy}{dx}+4y=0 \qquad \frac{dy}{dx}+y=0$$

Their respective solutions are

$$y(x)=C_1e^{-4x} \qquad y(x)=C_2e^{-x}$$

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If $(D+4)y=0$ then y is solution of $(D^2+5D+4)y=0$
Since the two solutions $y(x)=C_1e^{-4x}$ and $y(x)=C_2e^{-x}$ are linearly independent, a linear combination of them contains two arbitrary constants and so is the general solution :

$$y(x)=C_1e^{-4x}+C_2e^{-x}$$

Second-order linear equations with constant coeffs. and zero rhs (3)

The equation $D^2 + 5D + 4 = 0$ is called the auxiliary equation.

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Question : can we solve all second-order linear equations with constant coefficients and zero right-hand side by this method ?

Second-order linear equations with constant coeffs. and zero rhs (3)

The equation $D^2 + 5D + 4 = 0$ is called the auxiliary equation.

Question : can we solve all second-order linear equations with constant coefficients and zero right-hand side by this method ?

Let us assume that the equation can be written $(D-a)(D-b)y = 0$ with $a \neq b$. Then

$$y(x) = c_1 e^{ax} + c_2 e^{bx}$$

is the general solution of the equation.

Second-order linear equations with constant coeffs. and zero rhs (4)

<u>Special case</u> : equal roots of the auxiliary equation

In the case where tha auxiliary equation writes $(D-a)^2=0$, there is only one root and the previous theorem does not apply.
In this case the general solution writes :

$$y=(Ax+B)e^{ax}$$

Special case : complex conjugate roots of the auxiliary equation

Supose the roots of the auxiliary equation are $\alpha \pm i\beta$. These are unequated roots so the general solution of the equation is

$$y(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

This can be also re-written as

$$y(x) = e^{\alpha x}(c_3 \sin \beta x + c_4 \cos \beta x)$$

or,

$$y(x) = c_5 e^{\alpha x} \sin(\beta x + \gamma)$$

Second-order linear equations with constant coeffs. and zero rhs (6)

Example 1 : Solve the differential equation

$$y'' - 6y + 9y = 0$$

Second-order linear equations with constant coeffs. and zero rhs (6)

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The auxiliary equation writes

$$D^2 - 6D + 9 = 0$$

or, $(D-3)(D-3)y = 0$. There is only one root to the auxiliary equation so the general solution writes

$$y(x) = (Ax + B)e^{3x}$$

Example 2 : A mass m oscillates at the end of a spring where k is the spring constant.
The force exerted on the mass by the spring is proportional to displacement so Newton's second law writes

$$m\frac{d^2y}{dt^2} = -ky$$

which is also

$$D^2y + \omega^2 y = 0 \qquad\qquad \omega^2 = \frac{k}{m}$$

where $D = d/dt$. The roots of the auxiliary equation are $\pm i\omega$.
The solution can be written in any of the three forms :

$$\begin{aligned} y(x) &= Ae^{i\omega t} + Be^{-i\omega t} \\ &= c_1 \sin\omega t + c_2 \cos\omega t \\ &= c\sin(\omega t + \gamma) \end{aligned}$$

The object executes a simple harmonic motion.

- what if now the second-order equation is of the form

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

Second-order linear equations with constant coeffs. and non-zero rhs (1)

Example : Consider the equation

$$(D^2 + 5D + 4)y = \cos 2x$$

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We already know the solution of the equation with the right-hand side equal to zero :

$$y_c(x) = Ae^{-x} + Be^{-4x}$$

This solution is called the complementary function.

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We then have

$$(D^2 + 5D + 4)y_p = \cos 2x$$

$$(D^2 + 5D + 4)y_c = 0$$

so that

$$(D^2 + 5D + 4)(y_p + y_c) = \cos 2x + 0 = \cos 2x$$

One can then say that $y(x) = y_p + y_c$ is the general solution since it contains two independent arbitrary constants.

Second-order linear equations with constant coeffs. and non-zero rhs (2)

The general solution of an equation of the form

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

is

$$y = y_c + y_p$$

where the complementary function y_c is the general solution of the homogeneous equation and y_p is a particular solution.

Example : Consider

$$y'' - 6y' + 9y = 8e^x$$

Second-order linear equations with constant coeffs. and non-zero rhs (3)

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$\Rightarrow$ how can we find particular solutions in a more systematic way ?

Second-order linear equations with constant coeffs. and non-zero rhs (4)

$$y'' + y' - 2y = e^x$$

can be written as

$$(D-1)(D+2)y = e^x$$

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Let $u = (D+2)y$. The the differential equation becomes :

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$$ue^{-x} = \int e^{-x}e^x dx = x + c_1$$

$$u = xe^x + c_1 e^x$$

Second-order linear equations with constant coeffs. and non-zero rhs (5)

Then, the differential equation for y becomes

$$(D+2)y = xe^x + c_1 e^x$$

or,

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Second-order linear equations with constant coeffs. and non-zero rhs (5)

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We have obtained the general solution all in one process rather than finding the complementary function plus a particular solution in two separate processes.

Second-order linear equations with constant coeffs. and non-zero rhs (6)

Special cases :

- exponential rhs :

$$(D-a)(D-b)y = ke^{cx}$$

  (see example 5 p420)

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$$(D-a)(D-b)y = e^{cx}P_n(x)$$
where $P_n(x)$ is a n-order polynomial (see examples 7,8 p422).

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- exponential $\times$ polynomial rhs :
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where $P_n(x)$ is a n-order polynomial (see examples 7,8 p422).
- sine/cosine
$$(D-a)(D-b)y = k\cos\alpha x \qquad \text{or} \qquad (D-a)(D-b)y = k\sin\alpha x$$
first solve
$$(D-a)(D-b)y = ke^{i\alpha x}$$
and then take the real or imaginary part (see example 6 p420-421).

Laplace transform

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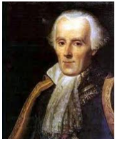
- The Laplace transform has the useful property that many relationships and operations over the originals $f(t)$ correspond to simpler relationships and operations over the images $F(p)$.
- It is named for Pierre-Simon Laplace, who introduced the transform in his work on probability theory.

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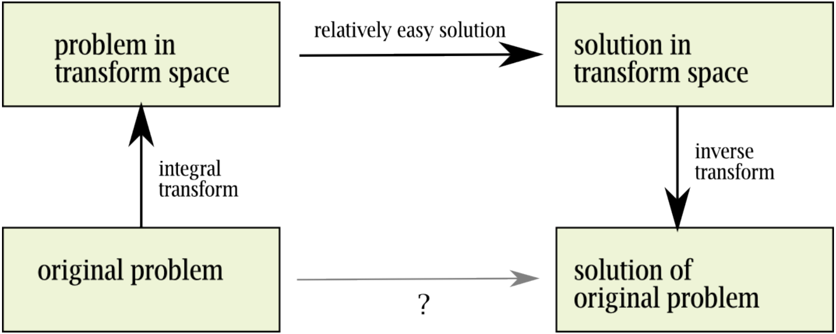
- the Fourier transform expresses a function or signal as a series of modes of vibration (frequencies), but the Laplace transform resolves a function into its moments.
- Like the Fourier transform, the Laplace transform is used for solving differential and integral equations.
- In physics and engineering it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems.

Laplace transform (3)



Laplace transform (4)

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Example 2 : Let $f(t) = e^{-at}$. Then

$$F(p) = \int_0^\infty e^{-at}e^{-pt} = \int_0^\infty e^{-(a+p)t} = \frac{1}{p+a} \qquad Re(p+a) > 0$$

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Laplace inverse transform

A table of transforms can be built-up and used to carry out the Laplace and inverse Laplace transforms (p469 in Boas)

Signal or Function	f(t)	F(s)
Impulse	$\delta(t)$	1
Step	$u(t)=1, \quad t \geq 0$	$\frac{1}{s}$
Ramp	$r(t)=t, \quad t \geq 0$	$\frac{1}{s^2}$
Exponential	$e^{-\alpha t} \quad e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
Damped Ramp	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$
Sine	$\sin(\beta t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\cos(\beta t)$	$\frac{s}{s^2+\beta^2}$
Damped Sine	$e^{-\alpha t}\sin(\beta t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped Cosine	$e^{-\alpha t}\cos(\beta t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Laplace inverse transform

A table of transforms can be built-up and used to carry out the Laplace and inverse Laplace transforms (p469 in Boas)

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There is little importance to these operations unless we can carry out the inverse transform, i.e.

$$\mathcal{L}(f(t)) = F(p) \qquad \rightarrow \qquad \mathcal{L}^{-1}(F(p)) = f(t)$$

Solution of diff. eqs. by Laplace transforms (1)

Let us consider $y(t)$ and look at $\mathcal{L}(y')$:

$$\mathcal{L}(y') = \int_0^\infty y'(t)e^{-pt}dt$$

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It is common to use the notation $Y = \mathcal{L}(y)$ so that

$$\begin{aligned} \mathcal{L}(y') &= -y(0) + pY \\ \mathcal{L}(y'') &= p^2 Y - py(0) - y'(0) \end{aligned}$$

Solution of diff. eqs. by Laplace transforms (2)

Example : Let us solve

$$y'' + 4y' + 4y = t^2 e^{-2t}$$

with initial conditions $y(t = 0) = 0$ and $y'(t = 0) = 0$.

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We take the Laplace transform of each term in the equation :

$$[p^2 Y - py(0) - y'(0)] + 4[-y(0) + pY] + 4Y = \mathcal{L}(t^2 e^{-2t})$$

One can show that

$$\mathcal{L}(t^2 e^{-2t}) = \frac{2}{(p+2)^3}$$

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and given the boundary (initial) conditions :

$$p^2 Y + 4pY + 4Y = \frac{2}{(p+2)^3}$$

or,

$$Y = \frac{2}{(p+2)^5}$$

We find that the inverse Laplace transform of Y is

$$y(t) = \frac{2t^4 e^{-2t}}{4!}$$

Solution of diff. eqs. by Laplace transforms (2)

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subject to initial conditions $y(0) = 10$, $y'(0) = 0$.

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We apply the same procedure :

$$p^2 Y - py(0) - y'(0) + 4Y = \mathcal{L}(\sin(2t))$$

or,

$$(p^2 + 4)Y - 10p = \frac{2}{p^2 + 4}$$

$$Y = \frac{10p}{p^2 + 4} + \frac{2}{(p^2 + 4)^2}$$

Using the Laplace formula table (L4 and L17) leads to

$$y = 10\cos 2t + \frac{1}{8}(\sin 2t - 2t\cos 2t)$$

Solution of diff. eqs. by Laplace transforms (2)

Example 3 : Let us solve

$$\begin{aligned} y' - 2y + z &= 0 \\ z' - y - 2z &= 0 \end{aligned}$$

subject to initial conditions $y(0) = 1$, $z(0) = 0$.

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Let us define $Y = \mathcal{L}(y)$ and $Z = \mathcal{L}(z)$ and take the Laplace transform of both equations :

$$\begin{aligned} pY - y(0) - 2Y + Z &= 0 \\ pZ - z(0) - Y - 2Z &= 0 \end{aligned}$$

or,

$$\begin{aligned} (p-2)Y + Z &= 1 \\ Y - (p-2)Z &= 0 \end{aligned}$$

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And since $z = 2y - y'$ we arrive to

$$z(t) = e^{2t} \sin t$$