Differentiaal Vergelijkingen In de Aardwetenschappen

Complex numbers - chapt 2

C. Thieulot (c.thieulot@uu.nl)

November 2015

new term, definition

Exercise for werkcollege

Homework

pay attention to this

Introduction (1)

- $\mathbb{P}$ prime numbers : 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,...

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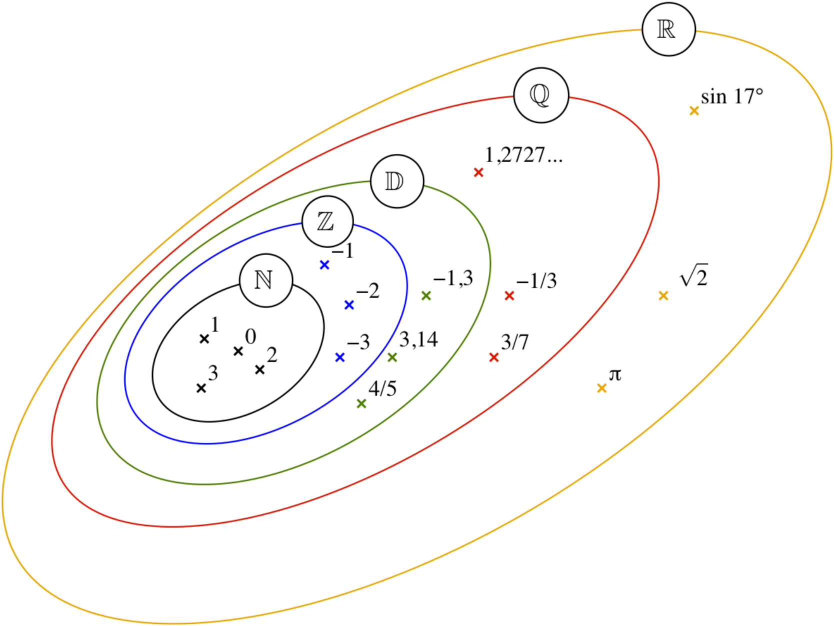
- $\mathbb{R}$ real numbers : $\sqrt{2}, \pi, e, 1/3, 0.12345, \ldots$
- $\mathbb{C}$ **complex numbers** : numbers which can be put in the form $x + iy$


ℝ
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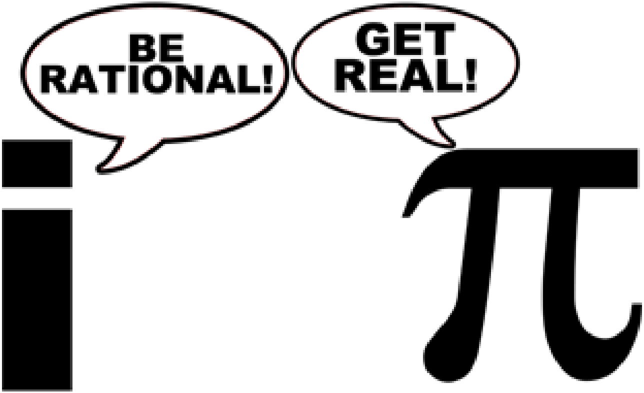
BE RATIONAL!
GET REAL!

Introduction (2)

Let us consider the quadratic equation :

$$az^2 + bz + c = 0$$

The solutions in $\mathbb{R}$ exist provided the discriminant $\Delta = b^2 - 4ac > 0$ and write

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Gerolamo Cardano, 1501-1576

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- Example 2 :
The solution of
$$z^2 - 2z + 2 = 0$$
is
$$z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm i$$

Real and imaginary parts of a complex number

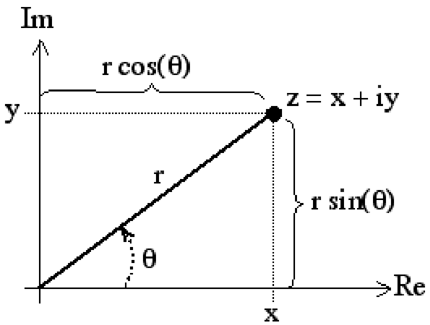
Let us look at a complex number :

$$z = 5 + 3i$$

- 5 is the real part
- 3 is the imaginary part
- if the real part is zero, z is pure imaginary
- complex numbers include both real numbers and pure imaginary numbers as special cases

The complex plane

The complex number z defines a point in the complex plane :



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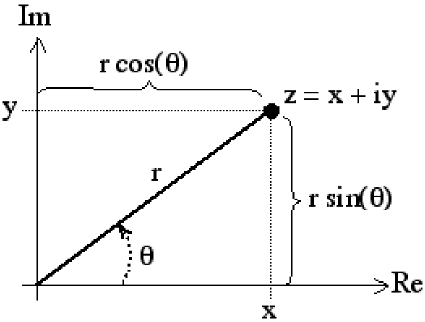
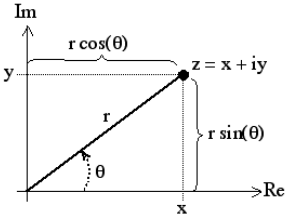


$$\cos\theta = x/r \qquad \sin\theta = y/r$$

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

Terminology, notation and algebra (1)



We have

$$z = x + iy = r(\cos\theta + i\sin\theta) = r\ e^{i\theta}$$

where

- r is called the modulus of z
- θ is called the angle of z

and

$$\mathrm{Re}\ z = x \qquad \mathrm{Im}\ z = y \qquad |z| = \mathrm{mod}\ z = r = \sqrt{x^2 + y^2}$$

Terminology, notation and algebra (2)

- The complex number $x - iy$ is called the complex conjugate of z.
- its notation is

$$\bar{z} = x - iy = r(\cos\theta - i\sin\theta) = r\ e^{-i\theta}$$

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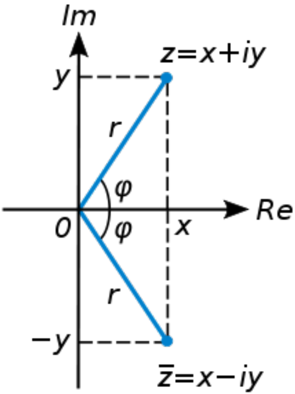

Terminology, notation and algebra (3)

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$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

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- Example 1 : compute the modulus of

$$\left| \frac{\sqrt{5} + 3i}{1 - i} \right| = \frac{|\sqrt{5} + 3i|}{|1 - i|} = \frac{\sqrt{14}}{\sqrt{2}} = \sqrt{7}$$

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- Example 2 : what are the real and imaginary parts of

$$\frac{2i-1}{i-2} = \frac{2i-1}{i-2}\,\frac{i+2}{i+2} = \frac{2i^2+4i-i-2}{i^2-4} = \frac{-4+3i}{-5} = \frac{4}{5} - \frac{3}{5}i$$

Terminology, notation and algebra (4)

- Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if both $x_1 = x_2$ and $y_1 = y_2$.

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$$(x + iy)^2 = (x^2 - y^2) + i(2xy)$$

leading to write

$$x^2 - y^2 = 0 \qquad \textit{and} \qquad 2xy = 2$$

which gives

$$x = y = 1 \qquad \textit{or} \qquad x = y = -1$$

an illustration of $z\overline{z} = x^2 + y^2 \in \mathbb{R}$

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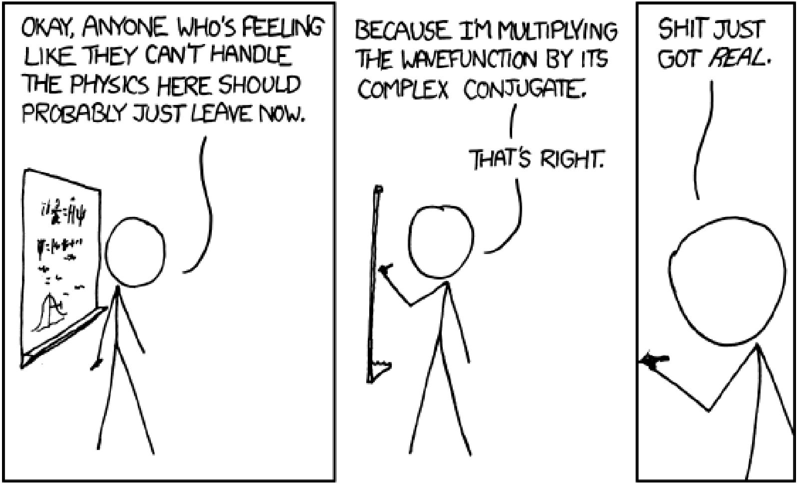
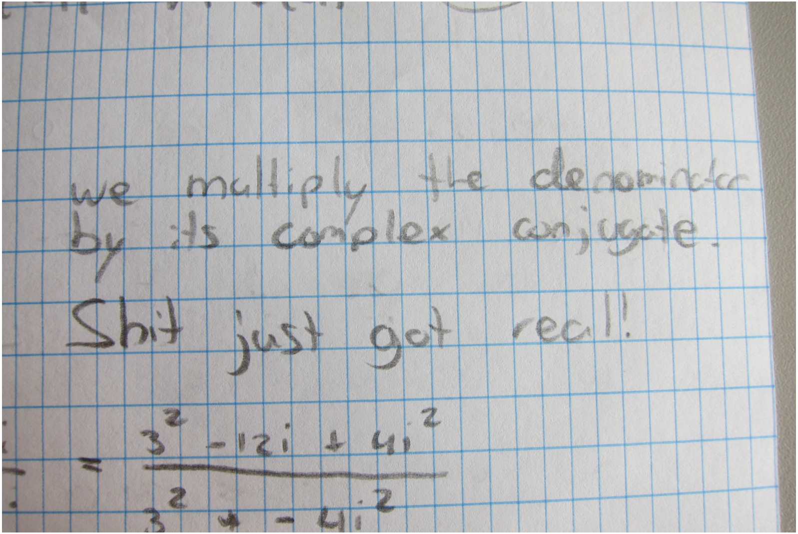


http://xkcd.com/



tentamen 2012.

Complex infinite series

Example 1 : Let us consider the following complex series :

$$1 + \frac{1+i}{2} + \frac{(1+i)^2}{4} + \frac{(1+i)^3}{8} + \dots \frac{(1+i)^n}{2^n} + \dots$$

Question : is this series absolutely convergent ?

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We use the ratio test :

$$\rho = \lim_{n\to\infty} \left| \frac{(1+i)^{n+1}}{2^{n+1}} \Big/ \frac{(1+i)^n}{2^n} \right| = \lim_{n\to\infty} \left| \frac{1+i}{2} \right| = \frac{\sqrt{2}}{2} < 1$$

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Since $\rho < 1$ the series is absolutely convergent.

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We can write further

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (re^{i\theta})^n$$

This is a geometric series with ratio $re^{i\theta}$ and therefore converges only if $|re^{i\theta}| < 1$, i.e. $r < 1$.

Complex power series (1)

From $\sum a_n x^n$ to $\sum a_n z^n$ where $a_n \in \mathbb{C}$ and $z \in \mathbb{C}$

- Example 1 :

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We use the ratio test to find for what z the series is absolutely convergent :

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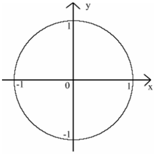

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This series converges for all values of z.

Complex power series (3)

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The series converges for $|z+1-i| < 3$. This is the interior of a disk of radius 3 and centered at $z = -1 + i$.

Elementary functions of complex numbers (1)

From $f(x), x \in \mathbb{R}$ to $f(z), z \in \mathbb{C}$ ...

Let us consider

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$$f(z) = \frac{z^2 + 1}{z - 3}$$

Then

$$f(i-2) = \frac{(i-2)^2 + 1}{(i-2) - 3} = \frac{4 - 4i}{-5 + i} = \frac{4 - 4i}{-5 + i}\frac{-5 - i}{-5 - i} = \frac{-12 + 8i}{13}$$

Elementary functions of complex numbers (2)

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&= e^z
\end{aligned}$$

Euler's formula

We have already established

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$

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We can write

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \end{aligned}$$

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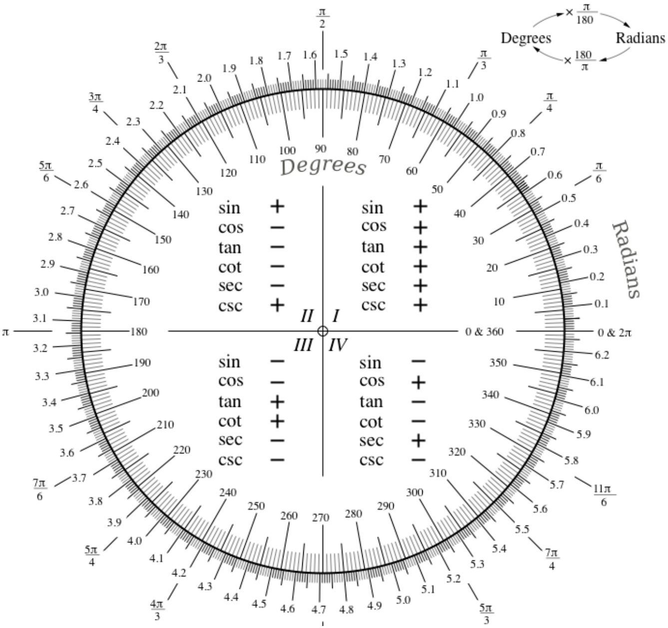
It also follows :

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$


Degrees
Radians
× π/180
× 180/π
Degrees
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π/4
π/6
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5π/3
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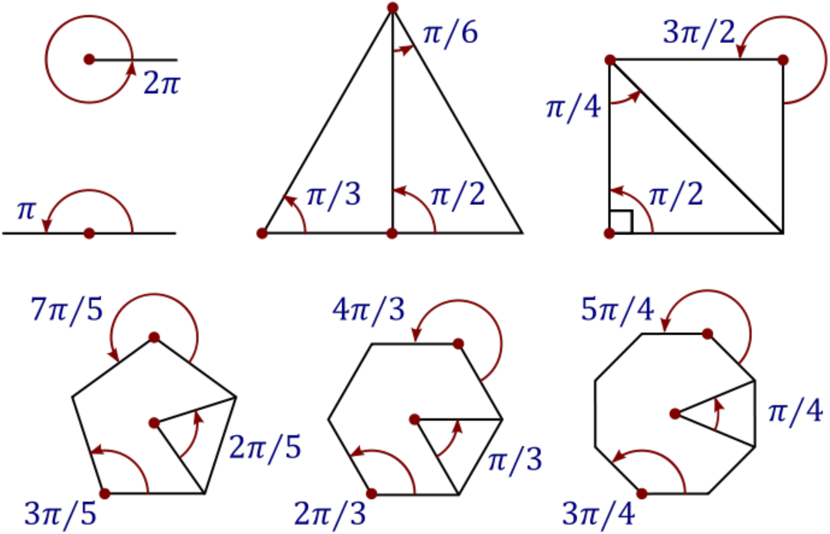

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4π/3
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2π/3
5π/4
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Complex numbers and angles

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$$e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(1+i)$$

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Complex numbers and angles

Let us start with :

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e^{i\pi} &= \cos\pi + i\sin\pi = -1 + i \cdot 0 = -1 \\
e^{3i\pi/2} &= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i \cdot (-1) = -i
\end{aligned}$$

Powers and roots of complex numbers

We have the following properties :

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

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$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$z^{1/n} = (r\ e^{i\theta})^{1/n} = r^{1/n}(\cos\theta + i\sin\theta)^{1/n} = r^{1/n}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$$

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$\rightarrow$ who wants to compute $(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10})^{25}$?
Instead, we write

$$z^{25} = (e^{i\pi/10})^{25} = e^{i25\pi/10} = e^{i5\pi/2} = e^{2i\pi}e^{i\pi/2} = 1 \cdot i = i$$

Example 2

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so

$$\begin{aligned} z^3 = 8 \quad &\Rightarrow \quad r^3 e^{3i\theta} = 8e^{2i\pi} \\ &\Rightarrow \quad r = 2, \quad \theta = 2n\pi/3 \end{aligned}$$

- $n = 0 \rightarrow z = 2e^0 = 2$
- $n = 1 \rightarrow z = 2e^{2i\pi/3}$
- $n = 2 \rightarrow z = 2e^{4i\pi/3}$
- $n = 3 \rightarrow z = 2e^{6i\pi/3} = 2e^{2i\pi} = 2$ (same as $n = 0$! !)

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Question : find and plot all values of $\sqrt[4]{-64}$

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- $n = 0 \to z = 2\sqrt{2}e^{i\pi/4}$
- $n = 1 \to z = 2\sqrt{2}e^{3i\pi/4}$
- $n = 2 \to z = 2\sqrt{2}e^{5i\pi/4}$
- $n = 3 \to z = 2\sqrt{2}e^{7i\pi/4}$

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$$-8i = 8 \cdot (-i) = 8e^{i(3\pi/2+2n\pi)}$$

$$\begin{aligned}\sqrt[6]{-8i} &= 8^{1/6} e^{i(3\pi/12+2n\pi/6)} \\ &= 2^{3/6} e^{i(\pi/4+n\pi/3)} \\ &= \sqrt{2} e^{i(\pi/4+n\pi/3)}\end{aligned}$$

Once again, the points are to be found by setting $n = 0, 1, 2, 3$.

Hyperbolic functions

Remember

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Taking $z = iy$ we get

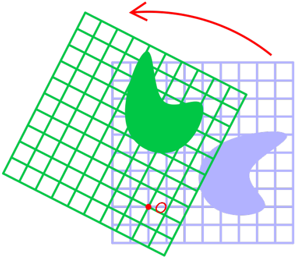
$$\sin iy = i\frac{e^{y} - e^{-y}}{2} \qquad \cos iy = \frac{e^{y} + e^{-y}}{2}$$

It is common to define so-called hyperbolic functions :

$$\sinh y = \frac{e^{y} - e^{-z}}{2} \qquad \cosh y = \frac{e^{y} + e^{-y}}{2}$$

complex numbers and rotations

- In geometry and linear algebra, a rotation is a transformation in a plane or in space that describes the motion of a rigid body around a fixed point.



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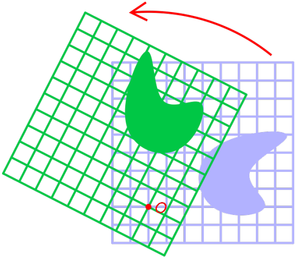

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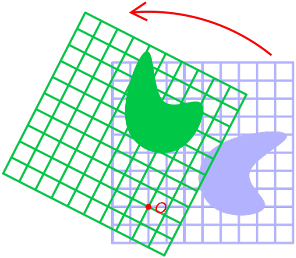

- Only a single angle is needed to specify a rotation in two dimensions the angle of rotation.
- To calculate the rotation two methods can be used, either *matrix algebra* or *complex numbers*. In each the rotation is acting to rotate an object counterclockwise through an angle θ about the origin.

Matrix algebra
To carry out a rotation using matrices the point (x, y) to be rotated is written as a vector, then multiplied by a matrix calculated from the angle, θ, like so :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

where (x, y) are the co-ordinates of the point after rotation, and the formulae for x and y can be seen to be

$$x' = x\cos\theta - y\sin\theta \quad (2)$$

$$y' = x\sin\theta + y\cos\theta. \quad (3)$$

complex numbers and rotations (2)

complex numbers

Points can also be rotated using complex numbers, as the set of all such numbers, the complex plane, is geometrically a two dimensional plane.
The point (x, y) in the plane is represented by the complex number

$$z = x + iy$$

This can be rotated through an angle θ by multiplying it by $e^{i\theta}$, then expanding the product using Euler's formula as follows :

$$\begin{aligned}
e^{i\theta} z &= (\cos\theta + i\sin\theta)(x + iy) && (4) \\
&= (x\cos\theta + iy\cos\theta + ix\sin\theta - y\sin\theta) && (5) \\
&= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta) && (6) \\
&= x' + iy', && (7)
\end{aligned}$$

which gives the same result as before,

$$\begin{aligned}
x' &= x\cos\theta - y\sin\theta && (8) \\
y' &= x\sin\theta + y\cos\theta. && (9)
\end{aligned}$$